

New Cosmological Singularities in Braneworld Models

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Abstract

Higher-dimensional braneworld models which contain both bulk and brane curvature terms in the action admit cosmological singularities of rather unusual form and nature. These ‘quiescent’ singularities, which can occur both during the contracting as well as the expanding phase, are characterised by the fact that while the matter density and Hubble parameter remain finite, all higher derivatives of the scale factor (\ddot{a} , $\ddot{\ddot{a}}$ etc.) diverge as the cosmological singularity is approached. The singularities are the result of the embedding of the (3+1)-dimensional brane in the bulk and can exist even in an empty homogeneous and isotropic (FRW) universe. The possibility that the present universe may expand into a singular state is discussed.

In this letter, we would like to draw attention to the fact that braneworld theory admits cosmological singularities of very unusual form and nature. The theory we work with in this paper is described by the action

$$S = M^3 \sum_i \left[\int_{\text{bulk}} (\mathcal{R} - 2\Lambda_i) - 2 \int_{\text{brane}} K \right] + \int_{\text{brane}} (m^2 R - 2\sigma) + \int_{\text{brane}} L(h_{\alpha\beta}, \phi), \quad (1)$$

the notation of which is standard. We use the signature and sign conventions of [1]. The sum in (1) is taken over the bulk components bounded by branes, and Λ_i is the cosmological constant in the i th bulk component. Note that we have included the curvature term in

the action for the brane with the coefficient m^2 . Such a term generically arises when one considers quantum effects generated by matter fields residing on the brane [2,3], in the spirit of the idea of the induced effective action [4,5]; its inclusion in the action is therefore mandatory. The lagrangian $L(h_{\alpha\beta}, \phi)$ corresponds to the presence of matter fields ϕ on the brane interacting with the induced metric $h_{\alpha\beta}$ and describes their dynamics. The extrinsic curvature $K_{\alpha\beta}$ on both sides of the brane is defined with respect to the inner normal n^a , as in [6–8].

Let us first consider the case with Z_2 symmetry of reflection, which requires equal cosmological constants on the two sides of the brane: $\Lambda_1 = \Lambda_2 = \Lambda$. The corresponding cosmological equation of theory (1) was derived in [2] and has the form

$$H^2 + \frac{\kappa}{a^2} = \frac{\rho + \sigma}{3m^2} + \frac{2}{\ell^2} \left[1 \pm \sqrt{1 + \ell^2 \left(\frac{\rho + \sigma}{3m^2} - \frac{\Lambda}{6} - \frac{C}{a^4} \right)} \right], \quad (2)$$

where the integration constant C corresponds to the presence of a black hole in the five-dimensional bulk solution, and the term C/a^4 (occasionally referred to as ‘dark radiation’) arises due to the projection of the bulk gravitational degrees of freedom onto the brane. The length scale ℓ is defined as

$$\ell = \frac{m^2}{M^3}. \quad (3)$$

The new singularities that we are going to discuss in this paper are connected with the fact that the expression under the square root of (2) *turns to zero* at some point during evolution, so that solutions of the cosmological equations *cannot be continued* beyond this point. There are essentially two types of ‘quiescent’ singularities displaying this behaviour:

A **type 1** singularity (S1) is essentially induced by the presence of the ‘dark radiation’ term under the square root of (2) and arises in either of the following two cases:

- $C > 0$ and the density of matter increases *slower* than a^{-4} as $a \rightarrow 0$. Such singularities occur if the universe is filled with matter having equation of state $P/\rho < 1/3$, an example is provided by pressureless matter (dust) for which $\rho \propto a^{-3}$. A special case is an empty universe ($\rho = 0$).

- The energy density of the universe is radiation-dominated so that $\rho = \rho_0/a^4$ and $C > \rho_0$.

The singularities discussed above can take place either in the past of an expanding universe or in the future of a collapsing one.

A **type 2** singularity (S2) arises if

$$\ell^2 \left(\frac{\sigma}{3m^2} - \frac{\Lambda}{6} \right) < -1. \quad (4)$$

In this case, it is important to note that the combination $\rho/3m^2 - C/a^4$ decreases monotonically as the universe expands. The expression under the square root of (2) can therefore become zero at suitably late times, in which case the cosmological solution cannot be extended beyond this time. S2 is even more interesting than S1 since: (i) it can occur during the late time expansion of the universe; (ii) it can occur *even if* dark radiation is entirely absent ($C = 0$).

For both S1 & S2, the scale factor $a(t)$ and its first time derivative remain finite, while all the higher time derivatives of $a(t)$ tend to infinity as the singularity is approached. As an example consider a type 2 singularity with $C = 0$, for which

$$\frac{d^n a}{dt^n} = \mathcal{O}([\rho(t) - \alpha]^{3/2-n}), \quad n \geq 2, \quad (5)$$

as $\rho(t) \rightarrow \alpha = \Lambda m^2/2 - \sigma - 3m^2/\ell^2$. We therefore find that the scalar curvature $R \rightarrow \infty$ near the singularity, while the energy density and pressure *remain finite*. Although this situation is quite unusual from the viewpoint of the intrinsic dynamics on the brane, it becomes comprehensible when one considers the embedding of the brane in the bulk. It is well known that the cosmological braneworld under consideration can be isometrically embedded in the five-dimensional solution of the vacuum Einstein equations described by the metric

$$ds^2 = -f_\kappa(r)dt^2 + \frac{dr^2}{f_\kappa(r)} + r^2 d\Omega_\kappa, \quad (6)$$

where

$$f_\kappa(r) = \kappa - \frac{\Lambda}{6}r^2 - \frac{C}{r^2}, \quad (7)$$

and $d\Omega_\kappa$ is the metric of the three-dimensional Euclidean space, sphere or pseudosphere corresponding to the value of $\kappa = 0, \pm 1$. The embedding of the brane is defined by the function $r = a(t)$, and one can then proceed to define evolution in terms of the proper cosmological time τ of the induced metric on the brane. The cosmological singularity under consideration is connected with the fact that the brane embedding is not extendable beyond some moment of time t because the function $a(t)$ that defines the embedding cannot be smoothly continued beyond this point (see Fig. 1).

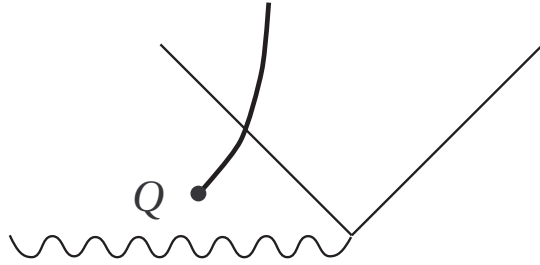


FIG. 1. Conformal diagram showing the trajectory of a spatially spherical braneworld embedded in the five-dimensional Schwarzschild space-time. The trajectory is not smoothly extendable beyond the point Q .

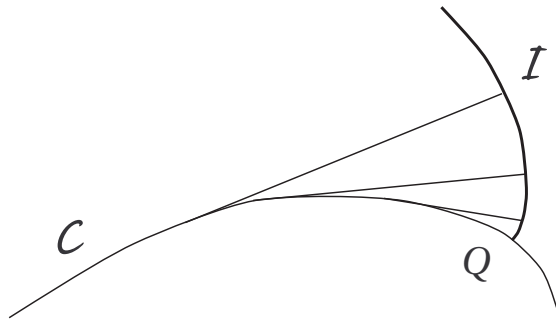


FIG. 2. The involute \mathcal{I} of a planar curve \mathcal{C} is not smoothly extendable beyond the starting point Q .

This specific feature of the brane embedding can be compared to the behaviour of the involute of a planar curve. The involute \mathcal{I} of a convex planar curve \mathcal{C} is a line which intersects the tangent lines of \mathcal{C} orthogonally [9]. \mathcal{I} can be visualised as the trajectory described by the end of a strained thread winding up from \mathcal{C} (see Fig. 2). The involute of a typical curve is sharp at the starting point Q so that it is not smoothly extendable beyond the point Q .

This analogy can be traced further. Note that the evolution of the brane in theory (1) is described by the following well-known equation:

$$m^2 G_{\alpha\beta} + \sigma h_{\alpha\beta} = T_{\alpha\beta} + M^3 \sum (K_{\alpha\beta} - h_{\alpha\beta} K) , \quad (8)$$

where the extrinsic curvature is summed over both sides of the brane. One can see that it is the influence of the sum of the extrinsic curvatures on the right-hand side that leads to the singularities under investigation so that the singularity of the Einstein tensor $G_{\alpha\beta}$ is accompanied by the singularity of the extrinsic curvature $K_{\alpha\beta}$, while the induced metric $h_{\alpha\beta}$ and the stress-energy tensor $T_{\alpha\beta}$ on the brane remain finite. Quite similarly, the involute of a curve is defined through the extrinsic curvature of its embedding in the plane, as is clear from Fig. 2, and its singularity at the point Q is connected with the fact that the extrinsic curvature diverges at this point. Specifically, the parametric equation for the involute $\mathbf{x}_*(s)$, $s \geq 0$, in Cartesian coordinates on the plane can be written as [9]

$$\mathbf{x}_*(s) = \mathbf{x}(q - s) + s \cdot \mathbf{x}'(q - s) , \quad (9)$$

where $\mathbf{x}(s)$ is the curve \mathcal{C} parametrized by the natural parameter s , and $\mathbf{x}(q) = \mathbf{x}_*(0)$ is the coordinate of the starting point Q of the involute. The extrinsic curvature of the involute is

$$k(s) = \frac{1}{s} , \quad (10)$$

which diverges at the starting point Q corresponding to $s = 0$.

One should also highlight an important difference between the 1D and 4D embeddings: the involute being one-dimensional, a singularity in its extrinsic curvature does not lead to a singularity in its intrinsic geometry. As we have seen, this is not the case with the brane

for which the extrinsic and intrinsic curvatures are related through (8), so that a singularity in $K_{\alpha\beta}$ is reflected in a singularity in $G_{\alpha\beta}$.

Interestingly, an S2 singularity can arise in the distant future of a universe resembling our own ! To illustrate this we rewrite Eq. (2) (with $C = 0$) as [8]

$$\frac{H^2(z)}{H_0^2} = \Omega_m(1+z)^3 + \Omega_\kappa(1+z)^2 + \Omega_\sigma + \underline{2\Omega_\ell \pm 2\sqrt{\Omega_\ell} \sqrt{\Omega_m(1+z)^3 + \Omega_\sigma + \Omega_\ell + \Omega_\Lambda}}, \quad (11)$$

The underlining emphasises those terms which cause the braneworld to differ from its general-relativistic counterpart. For simplicity, we shall only discuss the solution corresponding to the ‘+’ sign in (11) (called BRANE2 in [8]). In this case, it can be shown that our model satisfies the constraint equation

$$\Omega_m + \Omega_\kappa + \Omega_\sigma + \underline{2\sqrt{\Omega_\ell} \sqrt{1 - \Omega_\kappa + \Omega_\Lambda}} = 1, \quad (12)$$

where

$$\Omega_m = \frac{\rho_0}{3m^2 H_0^2}, \quad \Omega_\kappa = -\frac{\kappa}{a_0^2 H_0^2}, \quad \Omega_\sigma = \frac{\sigma}{3m^2 H_0^2}, \quad \Omega_\ell = \frac{1}{\ell^2 H_0^2}, \quad \Omega_\Lambda = -\frac{\Lambda}{6H_0^2}, \quad (13)$$

and the subscript ‘0’ refers to the present value of the various cosmological quantities.

Inequality (4) now becomes

$$\Omega_\sigma + \Omega_\ell + \Omega_\Lambda < 0, \quad (14)$$

and the limiting redshift, $z_s = a_0/a(z_s) - 1$, at which the braneworld becomes singular is given by

$$z_s = \left(-\frac{\Omega_\sigma + \Omega_\ell + \Omega_\Lambda}{\Omega_m} \right)^{1/3} - 1. \quad (15)$$

The time of occurrence of the singularity (measured from the present moment) can easily be determined from

$$T_s = t(z = z_s) - t(z = 0) = \int_{z_s}^0 \frac{dz}{(1+z)H(z)}, \quad (16)$$

where $H(z)$ is given by (11) (see also [10]). In Fig. 3 we show a specific braneworld model having $\Omega_m = 0.2$, $\Omega_\ell = 0.4$, $\Omega_\Lambda = \Omega_\kappa = 0$. In keeping with observations of high redshift

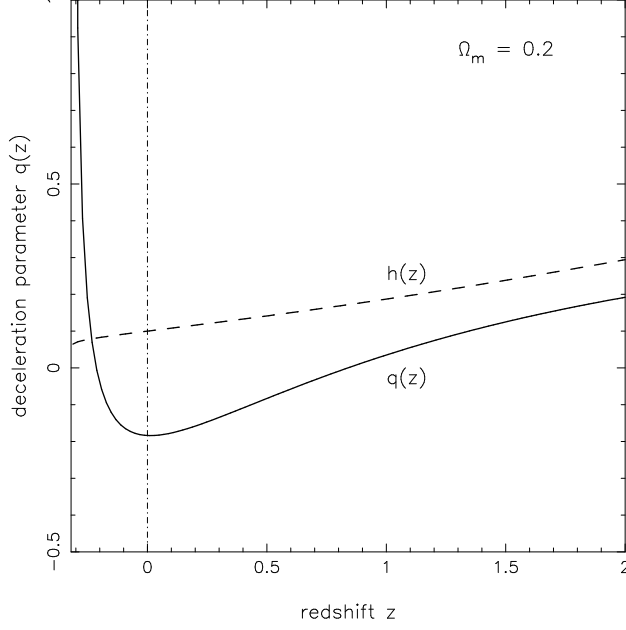


FIG. 3. The deceleration parameter (solid line) is shown for a braneworld model with $\Omega_m = 0.2$, $\Omega_\ell = 0.4$, $\Omega_\Lambda = \Omega_\kappa = 0$, and Ω_σ determined from (12). We find that $q(z) \rightarrow 0.5$ for $z \gg 1$ while $q(z) \rightarrow \infty$ as $z \rightarrow -0.312779\dots$. Currently $q_0 < 0$, which indicates that the universe is accelerating. Also shown is the dimensionless Hubble parameter $h(z) = 0.1 \times H(z)/H_0$ (dashed line) for this model. The vertical line at $z = 0$ shows the present epoch.

supernovae our model universe is currently accelerating [11], but will become singular at $z_s \simeq -0.3 \Rightarrow a(z_s) \simeq 1.4 \times a_0$, i.e. after $T_s \simeq 4.5 h_{100}^{-1}$ Gyr ($h_{100} = H_0/100$ km/sec/Mpc). Figure 3 demonstrates that the deceleration parameter becomes singular as z_s is approached:

$$q = -\frac{\ddot{a}}{aH^2} \equiv \frac{H'}{H}(1+z) - 1; \quad \lim_{z \rightarrow z_s} q(z) \rightarrow \infty, \quad (17)$$

while the Hubble parameter remains finite:

$$\frac{H^2(z_s)}{H_0^2} = \Omega_\ell - \Omega_\Lambda. \quad (18)$$

It should be noted that, for a subset of parameter values, inequality (14) can be satisfied simultaneously with $\Omega_\Lambda > \Omega_\ell$. In these models, the universe will recollapse (under the influence of the negative brane tension) before the S2 singularity is reached. (The marginal case $\Omega_\Lambda = \Omega_\ell$ corresponds to the Hubble parameter vanishing *at* the singularity.)

Now let us briefly consider the general braneworld without Z_2 symmetry. In this case, theory (1) leads to the following general cosmological equation for the brane embedded into the five-dimensional bulk [6]:

$$m^4 \left(H^2 + \frac{\kappa}{a^2} - \frac{\rho + \sigma}{3m^2} \right)^2 = 4M^6 \left(H^2 + \frac{\kappa}{a^2} - \frac{\Lambda_1 + \Lambda_2}{12} - \frac{C}{a^4} \right) - \frac{M^{12}}{36m^4} \left[\frac{\Lambda_1 - \Lambda_2 + E/a^4}{H^2 + \kappa/a^2 - (\rho + \sigma)/3m^2} \right]^2, \quad (19)$$

where C has the same meaning as in (2), E is another arbitrary integration constant, and Λ_1 and Λ_2 are the cosmological constants on the two sides of the brane. In this paper, we restrict ourselves to the situation where $\Lambda_1 = \Lambda_2 = \Lambda$, but $E \neq 0$. One should note that (19) reduces to (2) if $\Lambda_1 = \Lambda_2 = \Lambda$ and $E = 0$ [2,6].

It is easy to see that singularity S2 is always present in the past of the expanding brane. The reason for this rests in the *negative* character of the last term on the right-hand side of Eq. (19), which rapidly grows by absolute value as $a \rightarrow 0$, while the left-hand side of this equation is constrained to remain positive. In the case of an expanding brane (assuming $\Lambda_1 = \Lambda_2 = \Lambda$), the last term on the right-hand side of (19) rapidly decays and becomes unimportant. Therefore, provided (4) is satisfied, the expanding universe will encounter an S2 singularity in its future.

We note that all the singularities described above are absent in the limit $m \rightarrow 0$ which is frequently discussed in the literature. Indeed, in this limit, Eq. (19) takes the form

$$H^2 + \frac{\kappa}{a^2} = \frac{\Lambda_1 + \Lambda_2}{12} + \frac{C}{a^4} + \frac{(\rho + \sigma)^2}{36M^6} + \frac{M^6}{16} \left(\frac{\Lambda_1 - \Lambda_2 + E/a^4}{\rho + \sigma} \right)^2, \quad (20)$$

which only admits cosmological singularities associated with an infinite density of matter and dark radiation (C/a^4). These singularities are reached as $a \rightarrow 0$. (In general relativity, homogeneous and isotropic space-times generically admit only infinite-density singularities, whereas anisotropic space-times can be empty yet singular [12].)

Singularities of the kind discussed in this paper occur when the original equations of motion are non-linear with respect to the highest derivative. They have earlier been discussed in the context of Einstein gravity with the conformal anomaly [13]. (This result is not

surprising in view of the formal similarity between braneworld theory and GR-based models with the conformal anomaly, discussed in [8].) ‘Determinant singularities’ having a similar structure and properties are known to arise in the anisotropic Bianchi I model containing a dilaton coupled to a Gauss–Bonnet term in the action [14] (the general theory of such ‘determinant singularities’ is discussed in [15]). We should emphasise that both the past (S1) and future (S1, S2) singularities in the braneworld scenario occur for a wide range in parameter space and might provide an interesting alternative to the ‘big bang’/‘big crunch’ singularities of general relativity. In addition, the universe of Fig. 3 which terminates at the S2 singularity with $\rho_m \rightarrow \text{constant}$ as $t \rightarrow t_s$, provides an interesting contrast to the bleak finale presented by quintessence or cosmological-constant dominated models, in which $\rho_m \rightarrow 0$ as $t \rightarrow \infty$.

Finally, we would like to draw the reader’s attention to issues which we feel require further investigation. First of all, the singularities discussed in this paper occur for the very special class of homogeneous and isotropic spaces. The singularities in this case are characterized by the property that the energy density remains bounded while the curvature blows up as one approaches the space-time singularity. It is open to question whether singularities of this kind will persist in space-times having less symmetry, including the more general anisotropic and inhomogeneous case. Secondly, as the singularity is approached and the scalar curvature of the induced metric on the brane diverges, higher-order derivative terms in the effective action for the brane become important. Taking these into account may qualitatively modify the behaviour of solutions as is the case in general relativity. Thirdly, back-reaction associated with quantum effects such as particle production and vacuum polarization may influence the behaviour of cosmological solutions in the vicinity of the singularity. We hope to return to these issues in our subsequent work.

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